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Lyapunov 1-forms for flows

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Abstract. In this paper we find conditions which guarantee that a given flow Φ on a compact metric space X admits a Lyapunov 1-form ω lying in a prescribed Čech cohomology class $\xi \in \check{H}^1(X;\mathbb{R})$. These conditions are formulated in terms of the restriction of ξ to the chain recurrent set of Φ. The result of the paper may be viewed as a generalization of a well-known theorem by Conley about the existence of Lyapunov functions.

1. *Introduction*

Conley proved in [**1, 2**] that any flow $\Phi : X \times \mathbb{R} \to X$ on a compact metric space X decomposes into a chain recurrent flow and a gradient-like flow. More precisely, he proved the existence of a *Lyapunov function* for the flow, i.e. a continuous function $L : X \to \mathbb{R}$, which decreases along any orbit of the flow lying in the complement $X - R$ of the chain recurrent set $R \subset X$ of Φ and is constant on the connected components‖ of R.

THEOREM 1. (Conley [**1, 2**]) *Let $\Phi : X \times \mathbb{R} \to X$, $\Phi(x,t) = x \cdot t$, be a continuous flow on a compact metric space X. Then there exists a continuous function $L : X \to \mathbb{R}$, which is constant on the connected components of the chain recurrent set $R = R(\Phi)$ of the flow Φ and satisfies $L(x \cdot t) < L(x)$ for any $x \in X - R$ and $t > 0$.*

This important result led Conley to his programme of understanding very general flows as collections of isolated invariant sets linked by heteroclinic orbits.

‖ Conley (see [**2**, Theorem 3.6D]) proved that the chain transitive components of R coincide with the connected components of R.

Our aim in this paper is to go one step further and to analyze the flow within the chain recurrent set R, where it is typically complicated. As a new tool, we study the notion of a *Lyapunov 1-form* for Φ, which is a natural generalization of the notion of a Lyapunov function and has been introduced in a different context in Farber's papers [**5, 6**]. We prove that under some natural assumptions, a given Čech cohomology class $\xi \in \check{H}^1(X;\mathbb{R})$ can be represented by a continuous closed Lyapunov 1-form for the flow Φ.

The notion of a *continuous closed 1-form on a topological space* generalizes the notion of a continuous function. For the convenience of the reader, we recall the relevant definitions and the main properties of continuous closed 1-forms in §2, referring for more details to the papers [**5, 6**], where they were originally introduced. For the purpose of this introduction, let us say that continuous closed 1-forms are analogues of the familiar smooth closed 1-forms on differentiable manifolds. Any continuous closed 1-form ω on a topological space X canonically determines a Čech cohomology class $[\omega] \in \check{H}^1(X;\mathbb{R})$, which plays a role analogous to the de Rham cohomology class of a smooth closed 1-form. For any continuous curve $\gamma : [0,1] \to X$, the line integral $\int_\gamma \omega \in \mathbb{R}$ is defined and has the usual properties; in particular, it depends only on the homotopy class of the curve relative to the endpoints.

Definition 1. Consider a continuous flow $\Phi : X \times \mathbb{R} \to X$ on a topological space X. Let $Y \subset X$ be a closed subset invariant under Φ. A continuous closed 1-form ω on X is called a Lyapunov 1-form for the pair (Φ, Y) if it has the following two properties:

(L1) for every $x \in X - Y$ and every $t > 0$,

$$\int_x^{x\cdot t} \omega < 0,$$

where the integral is calculated along the trajectory of the flow;

(L2) there exists a continuous function $f : U \to \mathbb{R}$ defined on an open neighbourhood U of Y such that $\omega|_U = df$ and f is constant on any connected component of Y.

Any continuous function $L : X \to \mathbb{R}$ determines the closed 1-form $\omega = dL$ (see §2) and, in this special case, condition (L1) reduces to the requirement $L(x \cdot t) < L(x)$ for any $t > 0$ and $x \in X - Y$, while condition (L2) means that L is constant on any connected component of Y. Hence, for $\omega = dL$, Definition 1 reduces to the classical notion of a Lyapunov function, see [**13**].

The following remark illustrates Definition 1. Given a flow Φ on X and a Lyapunov 1-form ω for (Φ, Y) representing a non-zero Čech cohomology class $[\omega] = \xi \in \check{H}^1(X;\mathbb{R})$, the homology class $z \in H_1(X;\mathbb{Z})$ of any periodic orbit of Φ satisfies

$$\langle \xi, z \rangle \le 0$$

with equality if and only if the periodic orbit is contained in Y. Using this fact, one constructs flows such that no non-zero cohomology class $\xi \in \check{H}^1(X;\mathbb{R})$ contains a Lyapunov 1-form.

In this paper, we will associate with any cohomology class $\xi \in \check{H}^1(X;\mathbb{R})$ a subset $R_\xi \subset R$ of the chain recurrent set $R = R(\Phi)$ of the flow Φ, see §3 for details. The set R_ξ is closed and invariant under the flow and can be characterized as the projection of the

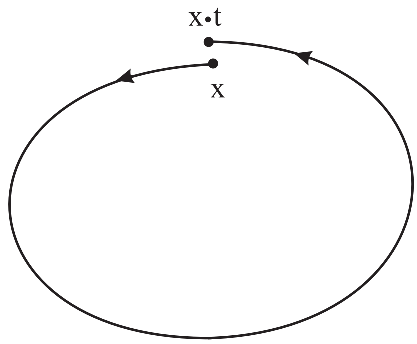

FIGURE 1. (δ, T)-cycle.

chain recurrent set of the natural lift of the flow to the Abelian cover of X associated with the class ξ.

A (δ, T)-*cycle* is a pair $(x, t) \in X \times \mathbb{R}$ satisfying $t \geq T$ and $d(x, x \cdot t) < \delta$. Here d denotes the distance function on X. See Figure 1. If X is locally path connected, any (δ, T)-cycle with small enough δ determines a closed loop, which first follows the flow line from x to $x \cdot t$ and then returns from $x \cdot t$ to x by a path contained in a suitably small ball. This leads to the notion of *a homology class* $z \in H_1(X; \mathbb{Z})$ *associated to a* (δ, T)-*cycle*, see Definition 4. The class z is uniquely defined if X is homologically locally 1-connected; without this assumption the homology class z associated with a (δ, T)-cycle might not be unique. The natural bilinear pairing $\langle \cdot, \cdot \rangle : \check{H}^1(X; \mathbb{R}) \times H_1(X; \mathbb{Z}) \to \mathbb{R}$ can be understood as $\langle \xi, z \rangle = \int_\gamma \omega$, where ω is a representative closed 1-form for $\xi \in \check{H}^1(X; \mathbb{R})$ and $\gamma : [0, 1] \to X$ is a loop representing the class $z \in H_1(X; \mathbb{Z})$. Despite the fact that the homology class z associated to a (δ, T)-cycle might depend (for wild X) on the choice of the connecting path between $x \cdot t$ and x, the construction is such that the value $\langle \xi, z \rangle \in \mathbb{R}$ only depends (for small enough $\delta > 0$) on the (δ, T)-cycle itself; see §3 for details.

The following theorem is our main result.

THEOREM 2. *Let Φ be a continuous flow on a compact, locally path-connected, metric space X and ξ a cohomology class in $\check{H}^1(X; \mathbb{R})$. Denote by C_ξ the subset*

$$C_\xi = R - R_\xi \tag{1.1}$$

of the chain recurrent set R of the flow Φ. Assume that the following two conditions are satisfied:

(A) $\xi|_{R_\xi} = 0$*; and*

(B) *there exist constants $\delta > 0$, $T > 1$, such that every homology class $z \in H_1(X; \mathbb{Z})$ associated to an arbitrary (δ, T)-cycle (x, t) with $x \in C_\xi$ satisfies*

$$\langle \xi, z \rangle \leq -1.$$

Then there exists a Lyapunov 1-form ω for (Φ, R_ξ) representing the cohomology class ξ. Moreover, the subset C_ξ is closed.

Conversely, if, for the given cohomology class ξ, there exists a Lyapunov 1-form for the pair (Φ, R_ξ) in the class ξ and if the set C_ξ is closed, then (A) and (B) hold true.

COROLLARY 1. *Let $\Phi : X \times \mathbb{R} \to X$ be a continuous flow on a compact, locally path-connected metric space. Any Čech cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$ satisfying $\xi|_R = 0$ (where $R = R(\Phi)$ denotes the chain recurrent set of the flow) contains a Lyapunov 1-form ω for (Φ, R).*

Corollary 1 follows directly from Theorem 2 since, under the assumption $\xi|_R = 0$, the set R_ξ coincides with R (compare Definition 5) and so the set C_ξ is empty. Corollary 1 also admits a simple, independent proof based on Conley's theorem 1.

COROLLARY 2. *Suppose $\Phi : X \times \mathbb{R} \to X$ is a flow on a compact locally path-connected metric space, whose chain recurrent set consists of finitely many rest points and periodic orbits. Then a Lyapunov 1-form for (Φ, Y), with suitable $Y \subset X$, exists in a non-trivial cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$ if and only if the homology classes of the periodic orbits are contained in the half-space*

$$H_\xi := \{z \in H_1(X; \mathbb{Z}) \mid \langle \xi, z \rangle \leq 0\}.$$

If this condition holds, then the set Y coincides with the union of the rest points and of those periodic orbits, for which the corresponding homology classes $z \in H_1(X; \mathbb{Z})$ satisfy $\langle \xi, z \rangle = 0$.

This corollary is a direct consequence of our main Theorem 2. The class of flows meeting its assumptions includes the Morse–Smale flows on closed manifolds.

Another interesting special case arises when $R_\xi = \emptyset$. From Theorem 2, we also deduce the following result.

COROLLARY 3. *Let $\Phi : X \times \mathbb{R} \to X$ be a continuous flow on a compact, locally path-connected, metric space X and let $\xi \in \check{H}^1(X; \mathbb{R})$ be a non-zero Čech cohomology class. The following two conditions are equivalent:*

(i) *$R_\xi = \emptyset$ and the flow satisfies condition (B) of Theorem 2;*

(ii) *there exists a Lyapunov 1-form for $(\Phi, \emptyset)$ representing the class ξ.*

If the class ξ is integral, i.e. $\xi \in \check{H}^1(X; \mathbb{Z})$, then either of these conditions is equivalent to the existence of a continuous locally trivial fibration $p : X \to S^1 \subset \mathbb{C}$ with the following properties. The function $t \mapsto \arg(p(x \cdot t))$ is differentiable, the derivative

$$\frac{d}{dt} \arg(p(x \cdot t)) < 0$$

is negative for all $x \in X$, $t \in \mathbb{R}$ and the cohomology class $p^*(\mu) \in \check{H}^1(X; \mathbb{Z})$ coincides with ξ, where $\mu \in \check{H}^1(S^1; \mathbb{Z})$ is the fundamental class of the circle S^1 oriented counterclockwise. In particular, for any angle $\theta \in S^1$, the set $K = p^{-1}(\theta)$ is a cross section of the flow Φ.

Recall that a closed subset $K \subset X$ is a *cross section* of the flow Φ if the flow map $K \times \mathbb{R} \to X$ is a surjective local homeomorphism (see Schwartzman [**11**]). A cross section K is transversal to the flow and the orbit of every point in X intersects K in forward and backward time.

We give a proof of Corollary 3 in §5.

Under slightly stronger assumptions, Schwartzman [**12**] proved, among other results, the equivalence of the property of having a cross section and the existence of a fibration $p : X \to S^1$ as stated in the second part of Corollary 3. In subsequent work, we will compare in detail our results with Schwartzman's beautiful paper [**11**].

The second part of Corollary 3 may be viewed as a generalization of Fried's results on the existence of cross sections to flows on manifolds [**9**, Theorem D]. The assumption of Fried to ensure the existence of cross sections is formulated in terms of the notion of the *homological directions of a flow*, which we now recall. A sequence $(x_n, t_n) \in X \times (1, \infty)$ is a *closing sequence based at* $x \in X$ if the sequences x_n and $x_n \cdot t_n$ tend to x. We will assume that X is a compact polyhedron. Then, any closing sequence determines uniquely a sequence of homology classes $z_n \in H_1(X; \mathbb{Z})$. Here z_n denotes the homology class of a loop, which starts at x_n, follows the flow until $x_n \cdot t_n$ and then returns to x_n along a 'short' path. Let D_X be the factor space $D_X = H_1(X; \mathbb{R})/\mathbb{R}_+$, where this space is topologized as the disjoint union of the unit sphere with the origin. Any closing sequence as described determines a sequence of 'homology directions' $\tilde{z}_n \in D_X$, the equivalence classes of z_n in D_X. The set of homology directions $D_\Phi \subset D_X$ of the flow Φ is defined as the set of all accumulation points of all sequences $\tilde{z}_n$ corresponding to closing sequences in X. As noted by Fried [**9**], it is enough to consider closing sequences (x_n, t_n) with $t_n \to \infty$.

PROPOSITION 1. *Let X be a finite polyhedron and let $\xi \in H^1(X; \mathbb{Z})$ be an integral cohomology class. Let $\Phi : X \times \mathbb{R} \to X$ be a continuous flow such that the chain recurrent set R_ξ is isolated in R. Then condition (B) of Theorem 2 is equivalent to Fried's condition that any homology direction $\tilde{z} = \lim \tilde{z}_n \in D_X$ of any closing sequence $(x_n, t_n) \in X \times (1, \infty)$ with $x_n \in C_\xi$ satisfies $\langle \xi, \tilde{z} \rangle < 0$.*

See §6 for a proof.

A comparison of Fried's results [**9**] with the results of this paper shows that our setting is more general in two respects: we allow spaces X of a more general nature and arbitrary real Čech cohomology classes ξ. In [**9**], X is required to be a compact manifold, possibly with a boundary, and the class ξ has to be integral. The equivalence of our condition (B) with Fried's condition [**9**, Proposition 1] holds only under these additional assumptions.

In [**5, 6**], two different generalizations, $\mathrm{cat}(X, \xi)$ and $\mathrm{Cat}(X, \xi)$, of the classical notion of the Lusternik–Schnirelman category $\mathrm{cat}(X)$ were introduced: here X is a finite polyhedron and $\xi \in H^1(X; \mathbb{R})$ a cohomology class. Using these new concepts, an extension of the Lusternik–Schnirelman theory for flows was constructed, see [**5, 6**]. The main results of [**5, 6**] allow us to estimate the number of fixed points of a flow under the assumption that (1) the fixed points are isolated in the chain recurrent set and (2) the flow admits a Lyapunov closed 1-form lying in the class ξ. The results of the present paper fit nicely in this programme and explain the nature of assumption (2). Note that (1) is similar in spirit (although formally not equivalent) to the property that the set R_ξ is isolated in the chain recurrent set R of the flow.

2. *Closed 1-forms on topological spaces*

In this section we recall the notion of a continuous closed 1-form on a topological space, which has been introduced in [**5, 6**].

Definition 2. Let X be a topological space. A *continuous closed 1-form* on X is defined by an open cover $\mathcal{U} = \{U\}$ of X and by a collection $\{\varphi_U\}_{U\in\mathcal{U}}$ of continuous functions $\varphi_U : U \to \mathbb{R}$ with the following property: for any two subsets $U, V \in \mathcal{U}$, the difference

$$\varphi_U|_{U\cap V} - \varphi_V|_{U\cap V} : U \cap V \to \mathbb{R} \tag{2.1}$$

is a locally constant function (i.e. constant on each connected component of $U \cap V$). Two such collections, $\{\varphi_U\}_{U\in\mathcal{U}}$ and $\{\psi_V\}_{V\in\mathcal{V}}$, are called equivalent if their union $\{\varphi_U, \psi_V\}_{U\in\mathcal{U}, V\in\mathcal{V}}$ satisfies condition (2.1). The equivalence classes are called continuous closed 1-forms on X.

Any continuous function $f : X \to \mathbb{R}$ (viewed as a family consisting of a single element) determines a continuous closed 1-form, which we denote by df (the differential of f).

A continuous closed 1-form ω vanishes, $\omega = 0$, if it is represented by a collection of locally constant functions.

The sum of two continuous closed 1-forms determined by collections $\{\varphi_U\}_{U\in\mathcal{U}}$ and $\{\psi_V\}_{V\in\mathcal{V}}$ is the continuous closed 1-form corresponding to the collection $\{\varphi_{U|U\cap V} + \psi_{V|U\cap V}\}_{U\in\mathcal{U}, V\in\mathcal{V}}$. Similarly, one can multiply continuous closed 1-forms by real numbers $\lambda \in \mathbb{R}$ by multiplying the corresponding representatives with λ. With these operations, the set of continuous closed 1-forms on X is a real vector space.

Continuous closed 1-forms behave naturally with respect to continuous maps: if $h : Y \to X$ is continuous and $\{\varphi_U\}_{U\in\mathcal{U}}$ determines a continuous closed 1-form ω on X, then the collection of continuous functions $\phi_U \circ h : h^{-1}(U) \to \mathbb{R}$ determines a continuous closed 1-form on Y which will be denoted by $h^*\omega$. As a special case of this construction, we will often use the operation of restriction of a closed 1-form ω to a given subset $A \subset X$: in this case h is the inclusion map $A \to X$ and the form $h^*\omega$ is simply denoted as $\omega|_A$.

A continuous closed 1-form ω on X can be integrated along continuous paths in X. Namely, let ω be given by a collection $\{\varphi_U\}_{U\in\mathcal{U}}$, and $\gamma : [0, 1] \to X$ be a continuous path. We may find a finite subdivision $0 = t_0 < t_1 < \cdots < t_N = 1$ of the interval $[0, 1]$ such that, for each $1 \le i \le N$, the image $\gamma([t_{i-1}, t_i])$ is contained in a single open set $U_i \in \mathcal{U}$. Then we define the line integral

$$\int_\gamma \omega := \sum_{i=1}^{N} \varphi_{U_i}(\gamma(t_i)) - \varphi_{U_i}(\gamma(t_{i-1})). \tag{2.2}$$

The standard arguments show that the integral (2.2) is independent of all choices and, in fact, depends only on the homotopy class of γ relative to its endpoints.

Consider the following exact sequence of sheaves over X,

$$0 \to \mathbb{R}_X \to C_X \to B_X \to 0, \tag{2.3}$$

where $\mathbb{R}_X$ is the sheaf of locally constant functions, C_X is the sheaf of real-valued continuous functions and B_X is the sheaf of germs of continuous functions modulo locally constant ones. More precisely, B_X is the sheaf corresponding to the presheaf $U \mapsto C_X(U)/\mathbb{R}_X(U)$. By the previous definitions, the global sections of the sheaf B_X are in one-to-one correspondence with continuous closed 1-forms on X. Hence, the space of all closed 1-forms on X is $H^0(X; B_X)$.

The exact sequence of sheaves (2.3) generates the cohomological exact sequence

$$0 \to H^0(X; \mathbb{R}_X) \to H^0(X; C_X) \xrightarrow{d} H^0(X; B_X) \xrightarrow{[\cdot]} H^1(X; \mathbb{R}_X) \to 0. \tag{2.4}$$

In this exact sequence, $H^0(X; C_X) = C(X)$ is the space of all continuous functions $f : X \to \mathbb{R}$ and the map d assigns to any continuous function f its differential $df \in H^0(X; B_X)$. The group $H^1(X; \mathbb{R}_X)$ is the Čech cohomology $\check{H}^1(X; \mathbb{R})$ (see [**14**, ch. 6]); the map $[\cdot]$ assigns to any closed 1-form ω its Čech cohomology class $[\omega] \in \check{H}^1(X; \mathbb{R})$. This proves the following lemma.

LEMMA 1. *A continuous closed 1-form $\omega \in H^0(X; B_X)$ equals df for some continuous function $f : X \to \mathbb{R}$ if and only if its Čech cohomology class $[\omega] \in \check{H}^1(X; \mathbb{R})$ vanishes, $[\omega] = 0$. Any Čech cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$ can be realized by a continuous closed 1-form on X.*

Note also that there is a natural homomorphism $\check{H}^1(X; \mathbb{R}) \to H^1(X; \mathbb{R})$ from Čech cohomology to singular cohomology. Using Lemma 1 and the well-known identification $H^1(X; \mathbb{R}) \simeq \mathrm{Hom}(H_1(X); \mathbb{R})$, it can be described as a pairing

$$\langle \cdot, \cdot \rangle : \check{H}^1(X; \mathbb{R}) \times H_1(X; \mathbb{Z}) \to \mathbb{R} \quad \text{where } \langle [\omega], [\gamma] \rangle = \int_\gamma \omega. \tag{2.5}$$

In other words, choosing a representative closed 1-form ω for a cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$ and a closed loop γ in X representing a homology class $z \in H_1(X; \mathbb{Z})$, the number $\langle \xi, z \rangle \in \mathbb{R}$ equals the line integral $\int_\gamma \omega$, which is independent of the choices.

Here is a generalization of the well-known *Tietze extension theorem*.

PROPOSITION 2. *Let X be a metric space and $A \subset X$ a closed subset. Let ω be a continuous closed 1-form on A, and let $\xi \in \check{H}^1(A; \mathbb{R})$ denote the Čech cohomology class of ω. Then, for any cohomology class $\xi' \in \check{H}^1(X; \mathbb{R})$ satisfying $\xi'|_A = \xi$, there exists a continuous closed 1-form ω' on X representing the cohomology class ξ', such that $\omega'|_A = \omega$.*

Proof. Choose an arbitrary continuous closed 1-form Ω' representing the class ξ'. Then $\Omega'|_A$ is cohomologous to ω, i.e. $\Omega'|_A - \omega = df$, where $f : A \to \mathbb{R}$ is a continuous function. By Tietze's extension theorem for functions, we find a continuous function $f' : X \to \mathbb{R}$ extending f. Then $\omega' = \Omega' - df'$ is a closed 1-form in the class ξ' satisfying $\omega'|_A = \omega$. □

The statement of Proposition 2 can be expressed as follows: a continuous closed 1-form ω on a closed subset $A \subset X$ can be extended to a continuous closed 1-form on X if and only if the cohomology class $[\omega] \in \check{H}^1(A; \mathbb{R})$ can be extended to a cohomology class lying in $\check{H}^1(X; \mathbb{R})$.

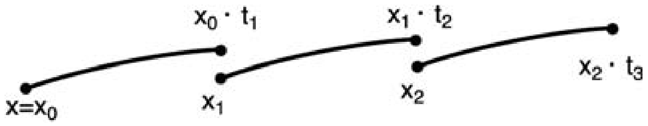

FIGURE 2. (δ, T)-chain.

3. *The chain recurrent set R_ξ*

The goal of this section is to introduce a new chain recurrent set $R_\xi = R_\xi(\Phi) \subset X$, which is associated with a flow Φ on X together with a Čech cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$. The set R_ξ appears in the statement of our main Theorem 2.

Throughout this section, we will assume that X is a locally path-connected compact metric space.

Recall that a space X is *locally path connected* if, for every open set $U \subset X$ and for every point $x \in U$, there exists an open set $V \subset U$ with $x \in V$ such that any two points in V can be connected by a path in U. Equivalently, X is locally path connected if and only if the path connected components of open subsets are open (see [**14**, p. 65]).

3.1. *Definition of R_ξ.*

Recall the definition of the chain recurrent set $R = R(\Phi)$ of the flow Φ. Given any $\delta > 0$, $T > 0$, a (δ, T)-*chain from $x \in X$ to $y \in X$* is a finite sequence $x_0 = x, x_1, \ldots, x_N = y$ of points in X and numbers $t_1, \ldots, t_N \in \mathbb{R}$ satisfying $t_i \geq T$ and $d(x_{i-1} \cdot t_i, x_i) < \delta$ for all $1 \leq i \leq N$. (See Figure 2.) Note that a (δ, T)-cycle (see §1) is a (δ, T)-chain of a special kind.

The *chain recurrent set* $R = R(\Phi)$ of the flow Φ is defined as the set of all points $x \in X$ such that, for any $\delta > 0$ and $T \geq 1$, there exists a (δ, T)-chain starting and ending at x. It is immediate from this definition that the chain recurrent set is closed and invariant under the flow and that R contains all fixed points and periodic orbits. The chain recurrent set R contains the set of all non-wandering points and, in particular, the positive and negative limit sets of any orbit [**1**, §II.6]. The set $R = R(\Phi)$ is a disjoint union of its *chain transitive components*†.

LEMMA 2. *Given a locally path-connected compact metric space X and a number $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that any two points $x, y \in X$ with $d(x, y) < \delta$ can be connected by a continuous path $\gamma : [0, 1] \to X$ contained in some open ε-ball.*

Proof. By the definition of local path connectedness, each point $x \in X$ has a neighbourhood V_x contained in the ε-ball B_x around x such that any two points in V_x can be connected by a path in B_x. Choose a finite subcover of the covering $\{V_x\}_{x \in X}$ of X and choose $\delta(\varepsilon)$ as the Lebesgue number of this finite cover. □

Definition 3. A pair (ε, δ) of real numbers $\varepsilon = \varepsilon(\xi) > 0$ and $\delta = \delta(\xi) > 0$ is called a scale of a non-zero cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$ if (1) $\xi|_B = 0$ for any ball $B \subset X$ of

† Recall that $x, y \in R$ belong to the same chain transitive component if, for any $\delta > 0$ and $T > 1$, there exist (δ, T)-chains from x to y and from y to x.

radius 2ε and (2) any two points $x, y \in X$ with $d(x, y) < \delta$ can be connected by a path in X contained in a ball of radius ε.

Such a scale always exists. In fact, we may realize the class ξ by a continuous closed 1-form $\omega = \{\phi_U\}_{U \in \mathcal{U}}$ with a finite open cover $\mathcal{U}$ and then take for ε half of the Lesbegue number of $\mathcal{U}$. Using Lemma 2, we may then find $\delta = \delta(\xi) > 0$ satisfying condition (2) in Definition 3.

We want to evaluate Čech cohomology classes on broken chains of trajectories of the flow which start and end at the same point. This can be done as follows. Let $\varepsilon = \varepsilon(\xi)$, $\delta = \delta(\xi)$ be a scale for $\xi \in \check{H}^1(X; \mathbb{R})$ (see Definition 3). Suppose we are given a *closed* (δ, T)-*chain*, i.e. a (δ, T)-chain from a point x to itself. We have a sequence of points $x_0 = x, x_1, \ldots, x_{N-1}, x_N = x$ of X and a sequence of numbers $t_1, \ldots, t_N \in \mathbb{R}$ with $t_i \geq T$, such that $d(x_{i-1} \cdot t_i, x_i) < \delta$ for any $1 \leq i \leq N$. We want to associate with such a chain a homology class $z \in H_1(X; \mathbb{Z})$. Choose continuous paths $\sigma_i : [0, 1] \to X$, where $1 \leq i \leq N$, connecting $x_{i-1} \cdot t_i$ with x_i and lying in a ball B_i of radius ε. We obtain a singular cycle which is a combination of the parts of the trajectories from x_{i-1} to $x_{i-1} \cdot t_i$ and the paths σ_i.

Definition 4. The homology class $z \in H_1(X; \mathbb{Z})$ of this singular cycle is said to be associated with the given closed (δ, T)-chain.

Note that the obtained class z may depend on the choice of paths σ_i (if the space X is wild, i.e. not locally contractible). However, the value

$$\langle \xi, z \rangle \in \mathbb{R}, \quad \text{where } \langle \xi, z \rangle = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_{i-1} \cdot t_i} \omega + \sum_{i=1}^{N} \int_{\sigma_i} \omega, \tag{3.1}$$

is independent of the paths σ_i. Indeed, if we use two different sets of curves σ_i and σ_i' then the difference of the corresponding expressions in (3.1) will be the integral over the sum of singular cycles $\sum_{i=1}^{N} (\sigma_i - \sigma_i')$, each cycle $\sigma_i - \sigma_i'$ being contained in a ball of radius 2ε. Since we know that the restriction of the cohomology class ξ on any such ball vanishes, we see that the right-hand side of (3.1) is independent of the choice of the curves $\sigma_1, \ldots, \sigma_N$.

The homology class $z \in H_1(X; \mathbb{Z})$ associated with a closed (δ, T)-chain is uniquely defined if X is homologically locally connected in dimension 1 (see [**14**, ch. 6, p. 340] for the definition) and $\delta > 0$ is sufficiently small.

Now we are ready to define the subset R_ξ of the chain recurrent set R.

Definition 5. Let $\varepsilon = \varepsilon(\xi)$ and $\delta = \delta(\xi)$ be a scale of a cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$. Then $R_\xi = R_\xi(\Phi)$ denotes the set of all points $x \in X$ with the following property: for any $0 < \delta' < \delta$ and $T > 1$, there exists a (δ', T)-chain from x to x such that $\langle \xi, z \rangle = 0$ for any homology class $z \in H_1(X; \mathbb{Z})$ associated with this chain.

Roughly, the set R_ξ can be characterized as the part of the chain recurrent set of the flow in which the cohomology class ξ does not detect the motion.

It is easy to see that R_ξ is closed and invariant with respect to the flow.

Note also that $R_\xi = R_{\xi'}$ whenever $\xi' = \lambda\xi$, with $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Thus, R_ξ depends only on the line through ξ in the real vector space $\check{H}^1(X; \mathbb{R})$.

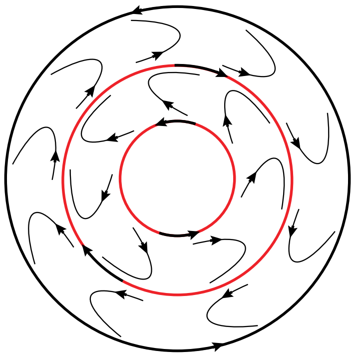

FIGURE 3. The flow on the planar ring Y.

Any fixed point of the flow belongs to R_ξ. The points of a periodic orbit belong to R_ξ if the homology class $z \in H_1(X; \mathbb{Z})$ of this orbit satisfies $\langle \xi, z\rangle = 0$.

Example. It may happen that the points of a periodic orbit belong to R_ξ although $\langle \xi, z\rangle \neq 0$ for the homology class z of the orbit. This possibility is illustrated by the following example.

Consider the flow on the planar ring $Y \subset \mathbb{C}$ shown on Figure 3. In polar coordinates (r, ϕ), the ring Y is given by the inequalities $1 \leq r \leq 3$ and the flow is given by the differential equations

$$\dot{r} = (r-1)^2(r-3)^2(r-5)^2, \quad \dot{\phi} = \sin\left(r \cdot \frac{\pi}{2}\right).$$

Let C_k, where $k = 1, 2, 3$, denote the circle $r = 2k-1$. The circles C_1, C_2, C_3 are invariant under the flow. The motion along the circles C_1 and C_3 has constant angular velocity 1. Identifying any point $(r, \phi) \in C_1$ with $(5r, \phi) \in C_3$, we obtain a torus $X = Y/\simeq$ and a flow $\Phi : X \times \mathbb{R} \to X$. The images of the circles $C_1, C_2, C_3 \subset Y$ represent two circles $C_1' = C_3'$ and C_2' on the torus X.

Let $\xi \in H^1(X; \mathbb{R})$ be a non-zero cohomology class which is the pullback of a cohomology class of Y. One verifies that, in this example, the set $R_\xi(\Phi)$ coincides with the whole torus X. In particular, $R_\xi(\Phi)$ contains the periodic orbits $C_1' = C_3'$ and C_2' although clearly $\langle \xi, [C_k']\rangle \neq 0$.

3.2. *R_ξ and dynamics in the free Abelian cover.* Now we will give a different characterization of R_ξ using the dynamics in the covering space associated with the class ξ. We will use Chapter 2 in [**14**] as a reference for notions related to the theory of covering spaces.

Recall our standing assumption that X is a locally path-connected compact metric space. For simplicity of exposition, we will additionally assume that X is connected.

Any Čech cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$ determines a homomorphism

$$h_\xi : \pi_1(X, x_0) \to \mathbb{R}, \quad h_\xi([\alpha]) = \langle \xi, [\alpha]\rangle = \int_\alpha \omega \in \mathbb{R}, \tag{3.2}$$

where $\alpha : [0, 1] \to X$ is a continuous loop $\alpha(0) = \alpha(1) = x_0$, $[\alpha] \in \pi_1(X, x_0)$ denotes its homotopy class and ω is a continuous closed 1-form in the class ξ. The map h_ξ is called *homomorphism of periods*.

The kernel of h_ξ is a normal subgroup $H = \mathrm{Ker}(h_\xi) \subset \pi_1(X, x_0)$. We want to construct a covering projection map $p_\xi : \tilde{X}_\xi \to X$, corresponding to H, i.e. $(p_\xi)_\# \pi_1(\tilde{X}_\xi, \tilde{x}_0) = H$. The uniqueness of such a covering projection map follows from [**14**, ch. 2, Corollary 3]. To show the existence, we may use [**14**, ch. 2, Theorem 13]; according to this theorem (see also Lemma 11 in [**14**, ch. 2]) we have to show that, for some open cover $\mathcal{U}$ of X, the subgroup $\pi_1(\mathcal{U}, x_0) \subset \pi_1(X, x_0)$ is contained in H. Here $\pi_1(\mathcal{U}, x_0) \subset \pi_1(X, x_0)$ denotes the subgroup generated by homotopy classes of the loops of the form $\alpha = (\gamma * \gamma') * \gamma^{-1}$ where γ' is a closed loop lying in some element of $\mathcal{U}$ and γ is a path from x_0 to $\gamma'(0)$. To show that this condition really holds in our situation, let us realize ξ by a closed 1-form $\omega = \{f_U\}_{U \in \mathcal{U}}$, $\mathcal{U}$ being an open cover of X. We claim that $\pi_1(\mathcal{U}, x_0) \subset H$ for this cover $\mathcal{U}$. Indeed, for any loop of the form $\alpha = (\gamma * \gamma') * \gamma^{-1}$ where γ' lies in some $U \in \mathcal{U}$,

$$\langle \xi, [\alpha] \rangle = \int_\alpha \omega = \int_{\gamma'} \omega = 0,$$

since γ' lies in U and $\xi|_U = 0$. Thus, *any Čech cohomology class* $\xi \in \check{H}^1(X; \mathbb{R})$ *uniquely determines a covering projection map* $p : \tilde{X}_\xi \to X$ *with connected total space* $\tilde{X}_\xi$*, such that* $(p_\xi)_\# \pi_1(\tilde{X}_\xi, \tilde{x}_0) = \mathrm{Ker}(h_\xi)$.

LEMMA 3. *Let X be a connected and locally path-connected compact metric space and let $\xi \in \check{H}^1(X; \mathbb{R})$ be a Čech cohomology class. The group of covering transformations of the covering map $\tilde{X}_\xi \to X$ is a finitely generated free Abelian group.*

Proof. The group of covering transformations of $\tilde{X}_\xi \to X$ can be identified with $\pi_1(X, x_0)/H$, which is isomorphic to the image of the homomorphism of periods $h_\xi(\pi_1(X, x_0)) \subset \mathbb{R}$. It is a subgroup of $\mathbb{R}$ and, hence, Abelian and it has no torsion. Therefore, it is enough to show that the image of the homomorphism of periods $h_\xi(\pi_1(X, x_0)) \subset \mathbb{R}$ is finitely generated.

Let $\omega = \{f_U\}_{U \in \mathcal{U}}$ be a continuous closed 1-form with respect to an open cover $\mathcal{U}$ representing ξ. Find an open cover $\mathcal{V}$ of X and a function $\kappa : \mathcal{V} \to \mathcal{U}$ such that, for any $V \in \mathcal{V}$, the set $U = \kappa(V) \in \mathcal{U}$ satisfies $\bar{V} \subset U$. We may realize ω with respect to the open cover $\mathcal{V}$ as $\omega = \{g_V\}_{V \in \mathcal{V}}$, where $g_V = f_U|_V$ for $U = \kappa(V)$. The path-connected components of open subsets of X are open (since X is locally path connected) and, hence, the family of path-connected components of the sets $V \in \mathcal{V}$ form an open cover of X. Using compactness, we may pass to a finite subcover; therefore, without loss of generality, we may assume that $\mathcal{V}$ is finite and the sets $V \in \mathcal{V}$ are path connected.

For any $V_1, V_2 \in \mathcal{V}$, the function $g_{V_1} - g_{V_2} : V_1 \cap V_2 \to \mathbb{R}$ is locally constant. We claim that the set $S_{V_1 V_2} \subset \mathbb{R}$ of real numbers $g_{V_1}(x) - g_{V_2}(x) \in \mathbb{R}$, where x varies in $V_1 \cap V_2$, is finite. Assume the contrary, i.e. there exists an infinite sequence $x_n \in V_1 \cap V_2$, where $n = 1, 2, \ldots$, such that

$$g_{V_1}(x_n) - g_{V_2}(x_n) \neq g_{V_1}(x_m) - g_{V_2}(x_m) \quad \text{for } n \neq m. \tag{3.3}$$

By compactness we may assume that x_n converges to a point $x_\infty \in X$. Denote $U_1 = \kappa(V_1), U_2 = \kappa(V_2)$, where $U_1, U_2 \in \mathcal{U}$. Then x_∞ belongs to $U_1 \cap U_2$ and, thus, $x_n \in U_1 \cap U_2$ for all large n. Let $W \subset U_1 \cap U_2$ denote the path-connected component of x_∞ in $U_1 \cap U_2$. Since W is open and contains x_∞, it follows that x_n belongs to W for all large enough n. The function $f_{U_1}(x) - f_{U_2}(x)$, where $x \in U_1 \cap U_2$, is continuous and locally constant; hence, it is constant for $x \in W$. We obtain that

$$f_{U_1}(x_n) - f_{U_2}(x_n) = f_{U_1}(x_m) - f_{U_2}(x_m)$$

for all large enough n and m. But this contradicts (3.3) since $f_{U_i}|_{V_i} = g_{V_i}$ and, therefore, $f_{U_i}(x_n) = g_{V_i}(x_n)$ for $i = 1, 2$ and all n.

The union

$$S = \bigcup_{V_1, V_2 \in \mathcal{V}} S_{V_1 V_2}$$

of all subsets $S_{V_1 V_2}$ is a finite subset of the real line. We will show now that the subgroup of $\mathbb{R}$ generated by S contains the group of periods $h_\xi(\pi_1(X, x_0))$. Let $\gamma : [0, 1] \to X$ be an arbitrary loop, $\gamma(0) = \gamma(1) = x_0$. We may find division points $t_0 = 0 < t_1 < t_2 < \cdots < t_N = 1$ and open sets $V_1, \ldots V_N \in \mathcal{V}$ such that $\gamma([t_{i-1}, t_i]) \subset V_i$ for $i = 1, 2, \ldots, N$. Then $\gamma(t_i) \in V_i \cap V_{i+1}$ for $i = 1, 2, \ldots, N$, where we understand that $V_{N+1} = V_1$. According to the definition of the line integral (see (2.2)), we have

$$\begin{aligned} h_\xi([\gamma]) = \langle \xi, [\gamma] \rangle &= \int_\gamma \omega \\ &= \sum_{i=1}^{N} [g_{V_i}(\gamma(t_i)) - g_{V_i}(\gamma(t_{i-1}))] \\ &= \sum_{i=1}^{N} [g_{V_i}(\gamma(t_i)) - g_{V_{i+1}}(\gamma(t_i))], \end{aligned}$$

which shows that any period $h_\xi([\gamma]) \in \mathbb{R}$ lies in the subgroup generated by the finite set $S \subset \mathbb{R}$. This implies that the group of periods is finitely generated and completes the proof of the lemma. □

LEMMA 4. *Assume that X is connected and locally path connected. Let ω be a continuous closed 1-form on X. Consider the covering map $p_\xi : \tilde{X}_\xi \to X$ determined by the Čech cohomology class $\xi = [\omega] \in \check{H}^1(X; \mathbb{R})$ of ω. Then $p_\xi^*(\omega) = dF$, where $F : \tilde{X}_\xi \to \mathbb{R}$ is a continuous function.*

Proof. Note that the integral $\int_\gamma \tilde{\omega} = 0$ vanishes for any closed loop γ in $\tilde{X}_\xi$, where $\tilde{\omega}$ denotes $p_\xi^*(\omega)$, by the construction of the covering $\tilde{X}_\xi$. Define

$$F(\tilde{x}) = \int_{\tilde{x}_0}^{\tilde{x}} \tilde{\omega}, \quad \tilde{x} \in \tilde{X}_\xi,$$

where $\tilde{x}_0 \in \tilde{X}_\xi$ is a base point and the integration is taken along an arbitrary path in $\tilde{X}_\xi$ connecting $\tilde{x}_0$ with $\tilde{x}$. It is easy to see that $F(\tilde{x})$ is independent of the choice of the path, F is continuous and $dF = p_\xi^*(\omega)$. □

The covering $\tilde{X}_\xi$ is now used to characterize the chain recurrent set R_ξ as follows.

PROPOSITION 3. *Let X be a connected and locally path-connected compact metric space. Given a continuous flow $\Phi : X \times \mathbb{R} \to X$ and a cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$, consider the free Abelian covering $p_\xi : \tilde{X}_\xi \to X$ associated with ξ (see earlier) and the canonical lift $\tilde{\Phi} : \tilde{X}_\xi \times \mathbb{R} \to \tilde{X}_\xi$ of the flow Φ to $\tilde{X}_\xi$. Fix a metric $\tilde{d}$ on $\tilde{X}_\xi$, which is invariant under the group of covering translations and such that the projection p_ξ is a local isometry. Then the chain recurrent set $R_\xi = R_\xi(\Phi) \subset X$ coincides with $p_\xi(R(\tilde{\Phi}))$, the image of the chain recurrent set $R(\tilde{\Phi}) \subset \tilde{X}_\xi$ of the lifted flow under the projection.*

Proof. Let $\varepsilon_0 > 0$ be such that, for any ball $\tilde{B} \subset \tilde{X}_\xi$ of radius ε_0 (with respect to the metric $\tilde{d}$), the following holds:
(1) $g\tilde{B} \cap \tilde{B} = \emptyset$ for any element $g \neq 1$ of the group of covering translations of $\tilde{X}_\xi$; and
(2) the projection p_ξ restricted to $\tilde{B}$ is an isometry.

We may satisfy (1) since the group of covering transformations of the covering $\tilde{X}_\xi$ acts properly discontinuously (see [**14**, p. 87]). Note that $\xi|_B = 0$, where $B = p_\xi(\tilde{B})$.

Let $\delta_0 = \delta(\varepsilon_0) > 0$ be the number given by Lemma 2. Then the pair $(\varepsilon_0, \delta_0)$ is a scale of ξ in the sense of Definition 3.

Suppose that a point $\tilde{x} \in \tilde{X}_\xi$ belongs to the chain recurrent set $R(\tilde{\Phi})$. Then, for any $\delta > 0$ and $T > 0$, there exists a (δ, T)-chain of the form $\tilde{x}_0 = \tilde{x}, \tilde{x}_1, \ldots, \tilde{x}_{N-1}, \tilde{x}_N = \tilde{x}$, $t_1, \ldots, t_N \in \mathbb{R}$, such that $\tilde{d}(\tilde{x}_{i-1} \cdot t_i, \tilde{x}_i) < \delta$ and $t_i \geq T$ for all $i = 1, 2, \ldots, N$. We will assume here that $\delta < \delta_0$. Projecting downstairs, we find a sequence

$$x_0 = x = p_\xi(\tilde{x}),\ x_1, \ldots, x_N = p_\xi(\tilde{x}) = x \in X,$$

with $x_i = p_\xi(\tilde{x}_i)$ satisfying $d(x_{i-1} \cdot t_i, x_i) < \delta$ for $i = 1, 2, \ldots, N$. This sequence forms a (δ, T)-chain in X which starts and ends at x. For any homology class $z \in H_1(X; \mathbb{Z})$ associated with this chain, one has $\langle \xi, z \rangle = 0$ since we can find a loop representing this class admitting a lift to the covering $\tilde{X}_\xi$. This shows that $p_\xi(R(\tilde{\Phi}))$ is contained in R_ξ.

To prove the inverse inclusion, assume that $\tilde{x} \in \tilde{X}$ is such that the point $x = p_\xi(\tilde{x}) \in X$ belongs to R_ξ. Hence, for any $\delta > 0$ and $T > 1$, we can find a (δ, T)-chain $x_0 = x, x_1, \ldots, x_N = x$, $t_i \in \mathbb{R}$, such that $d(x_{i-1} \cdot t_i, x_i) < \delta$, $t_i \geq T$ and, for any associated homology class $z \in H_1(X; \mathbb{Z})$, one has $\langle \xi, z \rangle = 0$. We will assume that $\delta < \delta_0$, where δ_0 is given as before. Choose continuous curves $\sigma_i : [0, 1] \to X$ such that $\sigma_i(0) = x_{i-1} \cdot t_i$ and $\sigma_i(1) = x_i$ for $i = 1, 2, \ldots, N$ and the image $\sigma_i([0, 1])$ is contained in a ball of radius ϵ_0. The concatenation of the parts of trajectories from x_{i-1} to $x_{i-1} \cdot t_i$ and the paths σ_i, where $i = 1, 2, \ldots, N$, forms a closed loop γ, which starts and ends at x. This loop lifts to a closed loop in the cover $\tilde{X}_\xi$ which starts and ends at $\tilde{x}$ since the homology class $z = [\gamma]$ of the loop satisfies $\langle \xi, z \rangle = 0$. The lift $\tilde{\gamma}$ of γ is a concatenation of parts of trajectories of the lifted flow $\tilde{\Phi}$ and the lifts $\tilde{\sigma}_i$ of the paths σ_i, where $i = 1, \ldots, N$. We obtain points $\tilde{x}_i \in \tilde{X}_\xi$, where $i = 0, 1, \ldots, N$, such that $p_\xi(\tilde{x}_i) = x_i$ and $\tilde{x}_0 = \tilde{x} = \tilde{x}_N$. Besides, we have $\tilde{\sigma}_i(0) = \tilde{x}_{i-1} \cdot t_i$, $\tilde{\sigma}_i(1) = \tilde{x}_i$ for $i = 1, 2, \ldots, N$. Since each σ_i lies in a ball of radius ϵ_0 in X, it follows from our assumption (1) that each path $\tilde{\sigma}_i$ lies in a ball $\tilde{B} \subset \tilde{X}_\xi$; from assumption (2) we find that $\tilde{d}(\tilde{x}_{i-1} \cdot t_i, \tilde{x}_i) < \delta$ for all $i = 1, \ldots, N$. Thus, we have found a (δ, T)-chain in $\tilde{X}_\xi$ starting and ending at $\tilde{x}$. This proves that ${p_\xi}^{-1}(R_\xi) \subset R(\tilde{\Phi})$, which is equivalent to $R_\xi \subset p_\xi(R(\tilde{\Phi}))$. $\square$

4. *Proof of Theorem 2*

In this section we will prove our main Theorem 2. The proof consists of two parts: the necessary conditions (easy) and the sufficient conditions (more difficult).

4.1. *Necessary conditions.*

If ω is a Lyapunov 1-form for (X, R_ξ), then, by Definition 1, $\omega|_U = df$ where f is a continuous function defined on an open neighbourhood $U \supset R_\xi$. Hence, the restriction of ξ on R_ξ vanishes, $\xi|_{R_\xi} = 0$, where $\xi \in \check{H}^1(X; \mathbb{R})$ denotes the cohomology class of ω. Thus, condition (A) in Theorem 2 is necessary. The following proposition implies that condition (B) of Theorem 2 is satisfied for any flow admitting a Lyapunov 1-form for (Φ, R_ξ) and having the property that the set $C_\xi = C_\xi(\Phi)$ is closed.

PROPOSITION 4. *Let $\Phi : X \times \mathbb{R} \to X$ be a continuous flow on a compact, locally path-connected, metric space X. Let ω be a Lyapunov 1-form for (Φ, Y), see Definition 1, where $Y \subset X$ is a closed flow-invariant subset. Let $C \subset X$ be a closed, flow-invariant subset such that $Y \cap C = \emptyset$. Then there exist numbers $\delta > 0$ and $T > 1$, such that any homology class $z \in H_1(X; \mathbb{Z})$ associated with any (δ, T)-cycle (x, t) with $x \in C$, satisfies*

$$\langle \xi, z \rangle \leq -1.$$

Here $\xi = [\omega] \in \check{H}^1(X; \mathbb{R})$ denotes the cohomology class of ω.

Proof. As $Y \cap C = \emptyset$, it follows that on C the function $x \mapsto \int_x^{x \cdot 1} \omega < 0$ is continuous and negative. Since C is compact, there exists a positive constant $c > 0$, such that

$$\int_x^{x \cdot 1} \omega < -c \quad \text{for all } x \in C. \tag{4.1}$$

Let (ε, δ) be a scale of the cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$, see Definition 3. Let $\eta > 0$ be such that, for any continuous curve $\sigma : [0, 1] \to X$ lying in a ball $B \subset X$ of radius ε, one has $\left|\int_\sigma \omega\right| < \eta$. We define

$$T = [(1 + \eta)/c] + 2, \tag{4.2}$$

where $[a]$ denotes the integer part of a number a. Unlike δ, the number T not only depends on the class ξ but on the chosen representative ω. We will show that $\delta > 0$ and $T > 1$ satisfy our requirements. Indeed, let (x, t) be a (δ, T)-cycle with $x \in C$ and let γ be a closed loop in X, obtained by first following the trajectory $x \cdot \tau$, where $\tau \in [0, t]$, and then returning from the endpoint $x \cdot t$ to x along a short path σ lying in a ball of radius ε. Then we have

$$\langle \xi, [\gamma] \rangle = \int_\gamma \omega = \int_x^{x \cdot t} \omega + \int_\sigma \omega.$$

For the second integral, we have $\int_\sigma \omega < \eta$ by construction. Since $t \geq T$ and T is an integer, we can estimate the first integral as

$$\int_x^{x \cdot t} \omega = \sum_{i=1}^{T} \int_{x \cdot (i-1)}^{x \cdot i} \omega + \int_{x \cdot T}^{x \cdot t} \omega < -Tc,$$

where we have used T multiplied by the inequality (4.1) and the estimate $\int_{x \cdot T}^{x \cdot t} \omega < 0$. By our choice of T, we have $Tc > 1 + \eta$ (cf. (4.2)) and so we see that $\langle \xi, z \rangle < -1$ for any homology class $z \in H_1(X; \mathbb{Z})$ associated with any (δ, T)-cycle in C. $\square$

4.2. *Constructing a Lyapunov 1-form: the first step.* In the remainder of this section, we prove the existence claim of Theorem 2. The proof is split into several lemmas. In a first step, we construct a Lyapunov 1-form for the flow restricted to C_ξ; later on, we will extend the obtained closed 1-form to a Lyapunov 1-form defined on the whole space X.

We start with the following combinatorial lemma.

LEMMA 5. *Any non-empty word w of arbitrary (finite) length in an alphabet consisting of L letters can be written as a product (concatenation)*

$$w = w_1 w_2 \cdots w_l$$

of $l \leq L$ non-empty words, such that in each word w_i, the first and last letters coincide.

Proof. We will use induction on L. For $L = 1$, our claim is trivial. We are left to prove the claim of the lemma assuming that it is true for words in any alphabet consisting of less than L letters. Consider a word w in an alphabet with L letters. Let a be the first letter of w. Finding the last appearance of a in w, we may write $w = w_1 w'$, where w_1 starts and ends with a and w' does not include a. We may now apply the induction hypothesis to w', which allows us to write $w' = w_2 w_3 \cdots w_l$, where $l \leq L$ and in each w_i the first and the last letters coincide. This clearly implies the lemma. □

Let ω be an arbitrary, continuous closed 1-form on X representing a cohomology class $\xi \in \check{H}^1(X; \mathbb{R})$. Our final goal will be to modify ω so that at the end we obtain a Lyapunov 1-form for (Φ, R_ξ).

LEMMA 6. *Under assumption (B) of Theorem 2, there exist $\mu > 0$ and $\nu > 0$ such that, for any $x \in C_\xi$ and $t \geq 0$, we have*

$$\int_x^{x \cdot t} \omega \leq -\mu t + \nu, \tag{4.3}$$

where the integral is calculated along the trajectory of the flow. In particular,

$$\lim_{t \to +\infty} \int_x^{x \cdot t} \omega = -\infty$$

and the convergence is uniform with respect to $x \in C_\xi$.

Proof. Let $\varepsilon > 0$ be such that, for any ball $B \subset X$ of radius ε, one has $\xi|_B = 0$ (see Definition 3) and for any continuous curve $\sigma : [0, 1] \to B$ holds $\left|\int_\sigma \omega\right| < \frac{1}{2}$. Let $\delta > 0$ be such that condition (B) of Theorem 2 holds for some $T > 1$ and, additionally, for any points $x, y \in X$ with $d(x, y) < \delta$, there is a continuous path σ connecting x and y and lying in a ball of radius ε. By (B),

$$\int_x^{x \cdot t} \omega < -\frac{1}{2}, \tag{4.4}$$

whenever $x \in C_\xi$, $t \geq T$ and $d(x, x \cdot t) < \delta$.

Using the compactness of X, we find a constant $M > 0$ such that, for any point $x \in X$ and any time $0 \leq t \leq T$,

$$\left| \int_x^{x \cdot t} \omega \right| < M. \tag{4.5}$$

Next we choose points $y_1, y_2, \dots, y_k$ in X, such that the open balls of radius $\delta/2$ with centres at these points cover X. Hence, for any $x \in X$, there exists an index $i \in \{1, \dots, k\}$, such that $d(x, y_i) < \delta/2$.

Given $x \in C_\xi$ and $t \geq 0$, we consider the sequence of points

$$x_j = x \cdot (jT) \in C_\xi, \quad j = 0, 1, \dots, N \quad \text{where } N = \left[\frac{t}{T}\right].$$

As previously explained, for any $j = 0, \dots, N$, there exists an index $1 \leq i_j \leq k$, such that

$$d(x_j, y_{i_j}) < \frac{\delta}{2}. \tag{4.6}$$

Thus, any point $x \in C_\xi$ determines a sequence of indices

$$i_0, i_1, \dots, i_N \in \{1, 2, \dots, k\}, \tag{4.7}$$

which, to a certain extent, encode the trajectory starting at x. If it happens that in the sequence (4.7) one has, for some $r < s$, $i_r = i_s$, then the part of the trajectory between $x_r = x \cdot (rT)$ and $x_s = x \cdot (sT)$ is a (δ, T)-cycle (in view of (4.6)) and by (4.4)

$$\int_{x_r}^{x_s} \omega < -\frac{1}{2}. \tag{4.8}$$

Let $1 \leq m \leq k$ be an index which appears in the sequence (4.7) most often. Clearly, it must appear at least $[(N+1)/k]$ times. Let α be the smallest number with $i_\alpha = m$ and β the largest number with $i_\beta = m$, so that $0 \leq \alpha < \beta \leq N$. Then, using (4.8), we have

$$\int_{x_\alpha}^{x_\beta} \omega \leq -\frac{N+1-k}{2k}. \tag{4.9}$$

To complete the argument, we need to estimate the remaining integrals $\int_x^{x_\alpha} \omega$ (corresponding to the beginning of the trajectory) and $\int_{x_\beta}^{x \cdot t} \omega$ (corresponding to the end of the trajectory).

View the sequence $i_0, i_1, \dots, i_\alpha$ as a word w in the alphabet $\{1, 2, \dots, k\}$ and apply Lemma 5. As a result, we may split the sequence $w = i_0, i_1, \dots, i_\alpha$ into $l \leq k$ subsequences $w_1, w_2, \dots, w_l$, each beginning and ending with the same symbol. If $w_j = i_r, i_{r+1}, \dots, i_s$ is one of the subsequences, where $r \leq s$, then $i_r = i_s$ and using (4.8) we find $\int_{x_r}^{x_s} \omega \leq 0$. In other words, the integral corresponding to each subsequence w_j is non-positive (in fact, it is less than $-\frac{1}{2}$ if the subsequence w_j has more than one symbol).

Now we want to estimate the contribution of the integrals corresponding to the word breaks in $w = w_1 w_2 \cdots w_l$. If w_j ends with the symbol i_s and the following subsequence w_{l+1} starts with i_{s+1}, then we have $\int_{x_s}^{x_{s+1}} \omega \leq M$ (see (4.5)). Thus, any word break contributes, at most, M to the integral.

The integral $\int_x^{x_\alpha} \omega$ is the sum of the contributions corresponding to the words w_j (which are all non-positive) and contributions of the word breaks (each is, at most, M). Since there are $l - 1 \leq k - 1$ word breaks, we obtain

$$\int_x^{x_\alpha} \omega \leq (l-1)M \leq (k-1)M. \tag{4.10}$$

Similarly, $\int_{x_\beta}^{x_N} \omega \leq (k-1)M$. For the remaining integral, we have $\int_{x_N}^{x\cdot t} \omega < M$, which again follows from (4.5), since $t - NT < T$.

Summing up, we finally obtain the estimate

$$\int_x^{x\cdot t} \omega < (k-1)M - \frac{N+1-k}{2k} + (k-1)M + M. \tag{4.11}$$

Hence, (4.3) holds true with the constants

$$\mu = \frac{1}{2kT} \quad \text{and} \quad \nu = (2k-1)M + \frac{1}{2}. \qquad \square$$

LEMMA 7. *Conditions (A) and (B) of Theorem 2 imply that the set $C_\xi = R - R_\xi$ is closed.*

Proof. Since $\xi|_{R_\xi} = 0$, we conclude that, for any continuous closed 1-form ω in the class ξ, the restriction $\omega|_{R_\xi}$ is the differential of a function and, hence, there exists a constant $C > 0$ such that, for any $x \in R_\xi$ and any $t > 0$,

$$\left| \int_x^{x\cdot t} \omega \right| < C. \tag{4.12}$$

Assume that the set C_ξ is not closed, i.e. there exists a sequence of points $x_n \in C_\xi$ converging to a point $x_0 \in R_\xi$. By Lemma 6, $\int_{x_n}^{x_n\cdot t} \omega < -\mu t + \nu$. Taking $t = t_0 = (\nu + 2C)/\mu$, we obtain

$$\int_{x_n}^{x_n\cdot t_0} \omega < -2C \tag{4.13}$$

for any $n = 1, 2, \ldots$. Passing to the limit with respect to n we find $\int_{x_0}^{x_0\cdot t_0} \omega \leq -2C$, contradicting the estimate (4.12). $\square$

LEMMA 8. *Let ω be a continuous closed 1-form on X realizing a class $\xi \in \check{H}^1(X;\mathbb{R})$. Assume that conditions (A) and (B) of Theorem 2 hold. Let $f : C_\xi \to \mathbb{R}$ be the function defined by*

$$f(x) := \sup_{t\geq 0} \int_x^{x\cdot t} \omega. \tag{4.14}$$

Then:

(i) *f is well defined and continuous;*

(ii) *$\omega_1 = \omega|_{C_\xi} + df$ is a continuous closed 1-form on C_ξ representing the cohomology class $\xi|_{C_\xi}$;*

(iii) *for any $x \in C_\xi$ and for any $t > 0$,*

$$\int_x^{x\cdot t} \omega_1 \leq 0, \tag{4.15}$$

i.e. ω_1 is a Lyapunov 1-form for the restricted flow $\Phi|_{C_\xi}$ in a weak sense;

(iv) *there exists a number $T > 1$, such that, for any $x \in C_\xi$ and any $t \geq T$,*

$$\int_x^{x\cdot t} \omega_1 \leq -1. \tag{4.16}$$

Proof. Let $t_0 > 0$ be the time such that $-\mu t_0 + \nu = 0$ with $\mu > 0$ and $\nu > 0$ as in Lemma 6. Then the supremum in (4.14) is achieved for $t \in [0, t_0]$ and, hence, we may write

$$f(x) = \max_{0 \le t \le t_0} \int_x^{x \cdot t} \omega.$$

The continuity of f now follows from the uniform continuity of the integral with respect to $(x, t) \in C_\xi \times [0, t_0]$.

Claim (ii) follows from (i). To prove (iii), we find that

$$\begin{aligned}\int_x^{x\cdot t} \omega_1 = \int_x^{x \cdot t} (\omega + df) &= \int_x^{x\cdot t} \omega + [f(x \cdot t) - f(x)] \\ &= \int_x^{x\cdot t} \omega + \sup_{\tau \ge 0} \int_{x \cdot t}^{x\cdot(t+\tau)} \omega - \sup_{\tau \ge 0} \int_x^{x \cdot \tau} \omega \\ &= \sup_{\tau \ge t} \int_x^{x\cdot\tau} \omega - \sup_{\tau \ge 0} \int_x^{x\cdot\tau} \omega \le 0.\end{aligned}$$

We next prove (iv). Let M denote the maximal value of the continuous function $f : C_\xi \to \mathbb{R}$ and m its minimal value. We apply Lemma 6 and obtain

$$\int_x^{x\cdot t} \omega_1 = \int_x^{x \cdot t} \omega + [f(x\cdot t) - f(x)] \le -\mu t + \nu + (M - m).$$

Hence, choosing T such that $-\mu T + \nu = -2 - (M - m)$, claim (iv) follows. □

4.3. *Second step: smoothing.* In this section, we describe a procedure for smoothing a continuous closed 1-form along the flow, which will be used in the proof of Theorem 2. It is a modification of a well-known method for continuous functions, see, for example, [**11**]. We use this construction to smooth the closed 1-form ω_1 which is constructed in the proof of Lemma 8.

A function $f : X \to \mathbb{R}$ is said to be differentiable along a continuous flow $\Phi : X \times \mathbb{R} \to X$ if the derivative $(d/dt) f(x \cdot t)|_{t=0}$ exists for any $x \in X$. More generally, a continuous closed 1-form ω on X is said to be *differentiable with respect to the flow* Φ if the derivative

$$\dot{\omega}(x) := \frac{d}{dt}\bigg(\int_x^{x\cdot t} \omega\bigg)\bigg|_{t=0} \tag{4.17}$$

exists for any $x \in X$. In this case, $\dot{\omega} : X \to \mathbb{R}$ is a function on X, which we call the *derivative of* ω *with respect to the flow* Φ. If ω is represented as $\omega = \{\varphi_U\}_{U \in \mathcal{U}}$ with respect to an open cover $\mathcal{U}$ of X, then, for $x \in U$ and t sufficiently small, we have $\int_x^{x \cdot t} \omega = \varphi_U(x \cdot t) - \varphi_U(x)$ and we see that a continuous closed 1-form is differentiable with respect to the flow Φ if and only if the local defining functions φ_U are.

LEMMA 9. *Assume that conditions (A), (B) of Theorem 2 hold. Then, there exists a continuous closed 1-form ω_2 on X in class $\xi \in \check{H}^1(X; \mathbb{R})$ with the following properties:*

(1) *ω_2 is differentiable with respect to the flow Φ;*

(2) *the derivative $\dot{\omega}_2 : X \to \mathbb{R}$ is a continuous function;*

(3) *for some $\sigma > 0$, one has $\dot{\omega}_2(x) \le -\sigma$ for all $x \in C_\xi$;*
(4) *$\omega_2|_U = 0$ for some open neighbourhood $U \subset X$ of R_ξ.*
In particular, $\omega_2|_{C_\xi}$ is a Lyapunov 1-form for the flow $\Phi|_{C_\xi}$.

Proof. Using assumptions (A), (B) and Lemma 7, we find a closed neighbourhood $V \subset X$ of R_ξ, such that $\xi|_V = 0$ and $V \cap C_\xi = \emptyset$. Here we use the continuity property of the Čech cohomology theory, see [**3**, ch. 10, Theorem 3.1].

Let ω_1 be the closed 1-form on C_ξ given by Lemma 8. Using Tietze's extension theorem (see Proposition 2), we may find a closed 1-form Ω_1 on X, such that $\Omega_1|_{C_\xi} = \omega_1$, $\Omega_1|_V = 0$, and $[\Omega_1] = \xi \in \check{H}^1(X; \mathbb{R})$.

Consider the covering map $p_\xi : \tilde{X}_\xi \to X$ corresponding to the Čech cohomology class ξ, see §3.2. By Lemma 4, we have $p_\xi^*(\Omega_1) = dF_1$, where $F_1 : \tilde{X}_\xi \to \mathbb{R}$ is a continuous function. Let $\tilde{C}_\xi$ denote the preimage $p_\xi^{-1}(C_\xi)$. Then, for any point $x \in \tilde{C}_\xi$ and $t \ge 0$, $F_1(x \cdot t) \le F_1(x)$, by Lemma 8(iii). Moreover, statement (iv) of Lemma 8 implies that there exists $T > 0$, such that

$$F_1(x \cdot t) - F_1(x) \le -1, \quad \text{for } x \in \tilde{C}_\xi, t \ge T. \tag{4.18}$$

Let $\varrho : \mathbb{R} \to [0, \infty)$ be a C^∞-smooth function with the following properties:
(a) the support of ϱ is contained in the interval $[-T-1, T+1]$;
(b) $\varrho|_{[-T,T]} = \text{constant} = \sigma > 0$;
(c) $\varrho(-t) = \varrho(t)$;
(d) $\varrho'(t) \ge 0$ for $t \le 0$;
(e) $\int_{\mathbb{R}} \varrho(t)\, dt = 1$.
Using ϱ, we define $F_2 : \tilde{X}_\xi \to \mathbb{R}$ by $F_2(x) = \int_{\mathbb{R}} F_1(x \cdot t)\varrho(t)\, dt$. It is clear that F_2 is continuous. Since $F_2(x \cdot s) = \int_{\mathbb{R}} F_1(x \cdot t)\varrho(t-s)\, dt$, we see that F_2 is differentiable with respect to the flow on $\tilde{X}_\xi$. If $x \in \tilde{C}_\xi$, we find, using (4.18) and the properties of ϱ, that

$$\begin{aligned} \left.\frac{dF_2(x \cdot s)}{ds}\right|_{s=0} &= -\int_{-T-1}^{T+1} F_1(x \cdot t)\varrho'(t)\, dt \\ &= \int_{-T-1}^{-T} [F_1(x \cdot (-t)) - F_1(x \cdot t)] \cdot \varrho'(t)\, dt \\ &\le -\int_{-T-1}^{-T} \varrho'(t)\, dt = -\sigma. \end{aligned} \tag{4.19}$$

Let G denote the group of covering transformations of the covering $\tilde{X}_\xi$. Using the homomorphism of periods (3.2), one sees that the class ξ determines a monomorphism $\alpha : G \to \mathbb{R}$, such that, for any $x \in \tilde{X}_\xi$ and any $g \in G$, we have

$$F_1(gx) - F_1(x) = \alpha(g). \tag{4.20}$$

Since $(gx) \cdot t = g(x \cdot t)$, we find $F_1((gx) \cdot t) = F_1(g(x \cdot t)) = F_1(x \cdot t) + \alpha(g)$ and multiplying by $\varrho(t)$ and integrating gives

$$F_2(gx) - F_2(x) = \alpha(g) \tag{4.21}$$

for any $x \in \tilde{X}_\xi$ and $g \in G$. Formula (4.21) states that the action of the covering translations changes F_2 by adding a constant and, therefore, F_2 determines a continuous closed 1-form on X. More precisely, $dF_2 = p_\xi^*(\omega_2)$ for some continuous closed 1-form ω_2 on X. Since F_2 is differentiable with respect to the flow on $\tilde{X}_\xi$ and the derivative $(d/ds)F_2(x \cdot s)$ is continuous (see (4.19)), the derivative $\dot{\omega}_2 : X \to \mathbb{R}$ is a well-defined, continuous function. Clearly, as Ω_1 vanishes on V, the form ω_2 vanishes on the open set $U \subset X$ of points $x \in X$ with $x \cdot [-T-1, T+1] \subset \operatorname{int} V$. Since $R_\xi \subset U$, this proves (iv).

Comparing (4.20) and (4.21), we find that the function $F_1 - F_2 : \tilde{X}_\xi \to \mathbb{R}$ is invariant under the covering translations. Hence, $F_1 - F_2 = f \circ p_\xi$, where $f : X \to \mathbb{R}$ is a continuous function. Therefore, $\Omega_1 - \omega_2 = df$, i.e. ω_2 lies in the cohomology class ξ. We know that ω_2 is differentiable with respect to the flow and $\dot{\omega}_2 \le -\sigma < 0$ on C_ξ. $\square$

4.4. *Third step: extension.* Now we complete the proof of the existence claim of Theorem 2.

Let $L : X \to \mathbb{R}$ be a Lyapunov function for (Φ, R). Such a function exists according to Theorem 1 of Conley [**2**]. We apply the smoothing procedure from the previous section to L. Namely, let $\rho : \mathbb{R} \to [0, \infty)$ be a C^∞-smooth function such that $\operatorname{supp}(\rho) = [-1, 1]$, $\int_{\mathbb{R}} \rho(t)\,dt = 1$, $\rho(-t) = \rho(t)$ and $\rho'(t) > 0$ for all $t \in (-1, 0)$ and set

$$L_1(x) = \int_{\mathbb{R}} L(x \cdot t)\rho(t)\,dt.$$

We find (precisely as in the previous section) that the derivative

$$\dot{L}_1(x) = \frac{d}{ds} L_1(x \cdot s)|_{s=0}$$

exists and is given by

$$\dot{L}_1(x) = \int_{-1}^{0} [L(x \cdot (-t)) - L(x \cdot t)]\rho'(t)\,dt. \tag{4.22}$$

This identity implies that $\dot{L}_1 : X \to \mathbb{R}$ is a continuous function and

$$\dot{L}_1(x) < 0 \quad \text{for any } x \in X - R.$$

Let ω_2 be the closed 1-form on X given by Lemma 9. We will set

$$\omega_3 = \omega_2 + \lambda(dL_1), \tag{4.23}$$

where $\lambda > 0$ and dL_1 is the differential of the function L_1 (see §2). In view of the construction of ω_2 and L_1, for any λ, the form ω_3 is a continuous closed 1-form on X representing the cohomology class ξ and satisfies condition (L2) of Definition 1.

We now show that, for λ large enough, ω_3 satisfies condition (L1) and, hence, it is a Lyapunov 1-form for (Φ, R_ξ). Indeed, ω_3 is differentiable along the flow and has the derivative $\dot{\omega}_3 = \dot{\omega}_2 + \lambda\dot{L}_1$. By Lemma 9, $\dot{\omega}_2 < 0$ on C_ξ. Hence, we may find an open neighbourhood W of C_ξ, so that $\dot{\omega}_2 < 0$ on W. By claim (iv) of Lemma 9, $\dot{\omega}_2 = 0$ vanishes on some open neighbourhood U of R_ξ, whereas $\dot{L}_1 < 0$ on $U - R_\xi$. Hence, we see that, for any $\lambda > 0$, the inequality $\dot{\omega}_3 < 0$ holds on W and on $U - R_\xi$. Finally, we shall show that $\dot{\omega}_3 < 0$ on $X - R_\xi$ for $\lambda > 0$ sufficiently large.

The function

$$x \mapsto -\frac{\dot{\omega}_2(x)}{\dot{L}_1(x)}, \quad x \in X - (U \cup W) \tag{4.24}$$

is well defined and continuous (recall that $\dot{L}_1 < 0$ on $X - R$). Since $X - (U \cup W)$ is compact, the function (4.24) is bounded. Choose $\lambda > 0$ to be larger than the maximum of (4.24). Then $\dot{\omega}_3(x) < 0$ holds for all $x \in X - R_\xi$, as desired.

This completes the proof of Theorem 2. □

These arguments prove the following, slightly stronger statement.

COROLLARY 4. *Under assumptions (A) and (B) of Theorem 2, there exists a continuous closed 1-form ω on X lying in the cohomology class $[\omega] = \xi \in \check{H}^1(X;\mathbb{R})$ which satisfies condition (L2) and the following stronger version of condition (L1): ω is differentiable with respect to the flow Φ (in the sense explained in §4.3), the derivative $\dot{\omega} : X \to \mathbb{R}$ is continuous and $\dot{\omega} < 0$ on $X - R_\xi$.*

5. *Proof of Corollary 3*

By Theorem 2, (i) implies (ii). Conversely, assume that there exists a Lyapunov 1-form ω for $(\Phi, \emptyset)$ representing the class ξ. Proposition 4 shows that condition (B) of Theorem 2 is satisfied. We are left to prove that $R_\xi = \emptyset$. Consider the covering $p_\xi : \tilde{X}_\xi \to X$ corresponding to the class ξ (see §3.2). It is enough to show that the chain recurrent set $R(\tilde{\Phi})$ of the lifted flow $\tilde{\Phi}$ in $\tilde{X}_\xi$ is empty (see Proposition 3). By Lemma 4, $p_\xi^*(\omega) = dF$, where $F : \tilde{X}_\xi \to \mathbb{R}$ is a continuous function. By assumption (ii), $F(x \cdot t) < F(x)$ for all $x \in \tilde{X}_\xi$ and $t > 0$. In particular, the function $\phi(x) = F(x \cdot 1) - F(x)$, defined on $x \in \tilde{X}_\xi$, is negative and invariant under the group of covering translations (see (4.20)); hence, it equals $\psi \circ p_\xi$, where $\psi : X \to \mathbb{R}$ is a continuous function on X. By the compactness of X, there exists $\sigma > 0$ such that $\phi(x) < -\sigma$ for all $x \in \tilde{X}_\xi$. Choose a metric d on $\tilde{X}_\xi$, which is invariant under the group of covering translations. There exists $\delta > 0$ such that, for any $x, y \in \tilde{X}_\xi$ with $d(x, y) < \delta$, one has $|F(x) - F(y)| < \sigma/2$. Now, assume that the points $x_0 = x, x_1, \ldots, x_N = x \in \tilde{X}_\xi$ and the numbers $t_1, \ldots, t_N \in \mathbb{R}$ represent a (δ, T)-chain with $T > 1$ of the lifted flow $\tilde{\Phi}$ in $\tilde{X}_\xi$, i.e. $t_i \geq T$ and $d(x_{i-1} \cdot t_i, x_i) < \delta$ for $i = 1, \ldots, N$. Then, for any $i = 1, 2, \ldots, N$, we have

$$F(x_{i-1} \cdot t_i) - F(x_{i-1}) < -\sigma \quad \text{and} \quad F(x_i) - F(x_{i-1} \cdot t_i) < \sigma/2,$$

which imply that $F(x_i) - F(x_{i-1}) < -\sigma/2$ and, hence, $F(x_N) - F(x_0) < 0$. The last inequality contradicts $x_0 = x = x_N$. This proves that there are no closed (δ, T)-chains in the covering $\tilde{X}_\xi$.

Thus we have shown that (i) and (ii) are equivalent.

Now assume that (ii) holds and the class ξ is integral, i.e. $\xi \in \check{H}^1(X;\mathbb{Z})$. As X is locally path connected and compact, it has finitely many path-connected components. Thus, without loss of generality, we may assume that X is path connected. Let ω be a Lyapunov 1-form for $(\Phi, \emptyset)$ satisfying the properties (iii) and (iv) of Corollary 4. Define a map $p : X \to S^1 \subset \mathbb{C}$ by choosing a point $x_0 \in X$ and setting

$$p(x) = \exp\left[2\pi i \int_{x_0}^{x} \omega\right],$$

where the line integral is taken along any path connecting x_0 with x. Since ξ is integral, the value $p(x) \in S^1$ does not depend on the choice of the path. The function

$$t \mapsto \arg(p(x \cdot t)) = 2\pi \int_{x_0}^{x \cdot t} \omega$$

is differentiable and the derivative

$$\frac{d}{dt} \arg(p(x \cdot t)) < 0$$

is negative. The equality $\xi = p^*(\mu)$ is immediate from the definition of p.

It remains to prove that p defines a locally trivial fibration. Pick $\eta \in \mathbb{R}$ and let $K_\eta = p^{-1}(\exp(2\pi i \eta)) \subset X$. For any $x \in K_\eta$, let $f_x : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f_x(0) = \eta$ and $p(x \cdot t) = \exp(2\pi i f_x(t))$ for all $t \in \mathbb{R}$. The function f_x is uniquely determined. It is differentiable and there exists $\varepsilon > 0$ such that, for all $x \in K_\eta$ and $t \in \mathbb{R}$,

$$\frac{d}{dt} f_x(t) < -\varepsilon.$$

Let $g_x = f_x^{-1}$ be the inverse function. Define $G : K_\eta \times \mathbb{R} \to X$ by $G(x,t) = x \cdot g_x(t)$. Then G is continuous and the diagram

$$\begin{tikzcd} K_\eta \times \mathbb{R} \arrow[rr, "G"] \arrow[dr, "e"'] & & X \arrow[dl, "p"] \\ & S^1 & \end{tikzcd} \tag{5.1}$$

commutes, where $e(x,t) = \exp(2\pi i t)$. This proves that p is a locally trivial fibration and that K_η is a cross section of the flow Φ. □

6. *Proof of Proposition 1*

Suppose that the set $C_\xi \subset R = R(\Phi)$ is closed but condition (B) of Theorem 2 is violated. Then, there exists a sequence of points $x_n \in C_\xi$ and numbers $t_n > 0$ such that the distances $d(x_n, x_n \cdot t_n)$ tend to 0 as $t_n \to \infty$ and

$$\langle \xi, z_n \rangle > -1, \tag{6.1}$$

where $z_n \in H_1(X; \mathbb{Z})$ denotes the homology class obtained by 'closing' the trajectory $x_n \cdot t$ for $t \in [0, t_n]$. Using the compactness of C_ξ, we may additionally assume that x_n converges to a point $x \in C_\xi$. Since we assume that the class ξ is integral, we may rewrite (6.1) in the form $\langle \xi, z_n \rangle \geq 0$. Thus, we obtain a closing sequence (x_n, t_n) such that for any homology direction $\tilde{z} \in D_X$ associated with it, $\langle \xi, \tilde{z} \rangle \geq 0$, i.e. the condition in Fried [**9**] also fails to hold.

Conversely, we now show that the condition of Fried [**9**] holds assuming that condition (B) of Theorem 2 is satisfied. Fix a norm $\|\|$ on the vector space $H_1(X; \mathbb{R})$. As X is a polyhedron, there exists $\delta > 0$ so that for any $\delta/2$-ball B in X the inclusion $B \to X$ is null-homotopic. Furthermore, there exists a constant $C > 0$ such that, for any homology class $z \in H_1(X; \mathbb{Z})$ associated with a (δ, T)-cycle (x, t) in X, one has

$$\|z\| \leq Ct. \tag{6.2}$$

Let ω be a continuous closed 1-form in the class ξ. By Lemma 6, there exist $\mu > 0$ and $\nu > 0$ such that

$$\int_x^{x\cdot t} \omega \le -\mu t + \nu \tag{6.3}$$

for all $x \in C_\xi$ and $t > 0$. Let $\eta > 0$ be such that $\left|\int_\gamma \omega\right| < \eta$ for any curve lying in a ball of radius $\delta/2$. Estimate (6.3) implies that

$$\langle \xi, z\rangle \le -\mu t + \nu + \eta \tag{6.4}$$

for any (δ, T) cycle (x, t) with $x \in C_\xi$, where $z \in H_1(X; \mathbb{Z})$ denotes the associated homology class. Since $\langle \xi, z\rangle \ge -c\|z\|$, where $c > 0$, we obtain that the homology class z of any (δ, T)-cycle (x, t) with $x \in C_\xi$ satisfies

$$\|z\| \ge \frac{\mu}{c}\cdot t - \frac{\nu+\eta}{c}. \tag{6.5}$$

Now, let (x_n, t_n) be a closing sequence (as defined in §1), where $x_n \in C_\xi$, such that x_n converges to a point $x \in C_\xi$ and $t_n \to \infty$. Let $z_n \in H_1(X; \mathbb{Z})$ denote the homology class determined by closing (x_n, t_n). Then (6.5) implies that $\|z_n\| \to \infty$. By (6.2), $-t \le -\|z\|/C$ which, when substituted into (6.4), leads to

$$\left\langle \xi, \frac{z_n}{\|z_n\|}\right\rangle \le -\frac{\mu}{C} + \frac{\nu+\eta}{\|z_n\|}.$$

Therefore, we obtain for the homology direction $z_n/\|z_n\| \in D_X$ of the class z_n the estimate

$$\left\langle \xi, \frac{z_n}{\|z_n\|}\right\rangle \le -\frac{\mu}{2C} < 0,$$

if n is large. This shows that Fried's condition [**9**] is satisfied. □

7. *Examples*

Example 1. Here we describe a class of examples of flows $\Phi : X \times \mathbb{R} \to X$, for which there exists a cohomology class ξ satisfying the conditions (A) and (B) of Theorem 2.

Let M be a closed smooth manifold with a smooth vector field v. Let $\Psi : M \times \mathbb{R} \to M$ be the flow of v. Assume that the chain recurrent set $R(\Psi)$ is a union of two disjoint closed sets $R(\Psi) = R_1 \cup R_2$, where $R_1 \cap R_2 = \emptyset$. Out of these data, we will construct a flow Φ on

$$X = M \times S^1$$

such that $R_\xi(\Phi) = R_1 \times S^0$, $C_\xi = R_2 \times S^1$. Here $\xi \in H^1(X; \mathbb{Z})$ denotes the cohomology class induced by the projection onto the circle $X \to S^1$ and $S^0 \subset S^1$ is a two-point set.

Let $\theta \in [0, 2\pi]$ denote the angle coordinate on the circle S^1. We will need two vector fields w_1 and w_2 on S^1, $w_1 = \cos(\theta)\cdot \partial/\partial\theta$ and $w_2 = \partial/\partial\theta$. The field w_1 has two zeros $\{p_1, p_2\} = S^0 \subset S^1$ corresponding to the angles $\theta = \pi/2$ and $\theta = 3\pi/2$.

Let $f_i : M \to [0, 1]$, where $i = 1, 2$, be two smooth functions having disjoint supports and satisfying $f_1|_{R_1} = 1$, $f_2|_{R_2} = 1$.

Consider the flow $\Phi : X \times \mathbb{R} \to X$ determined by the vector field

$$V = v + f_1 w_1 + f_2 w_2.$$

Any trajectory of V has the form $(\gamma(t), \theta(t))$, where $\dot\gamma(t) = v(\gamma(t))$, i.e. $\gamma(t)$ is a trajectory of v. It follows that the chain recurrent set of V is contained in $R(\Psi) \times S^1$. Over R_1 we have the vertical vector field w_1 along the circle which has two points $S^0 \subset S^1$ as its chain recurrent set. Over R_2 we have the vertical vector field w_2 which has all of S^1 as the chain recurrent set. We see that $R_1 \times S^0 = R_\xi(\Phi)$, $R_2 \times S^1 = C_\xi$. Hence, $\xi|_{R_\xi} = 0$ (and C_ξ is closed). Clearly condition (B) of Theorem 2 is satisfied as well.

Example 2. Let $X = T^2$, thought of as $\mathbb{R}^2/\mathbb{Z}^2$ with coordinates x and y on $\mathbb{R}^2$. Any cohomology class $\xi \in H^1(T^2; \mathbb{R})$ can be written as $\xi = \mu[dx] + \nu[dy]$, where dx and dy are the standard coordinate 1-forms. We consider the flow of the following vector field

$$V = f(x, y) \cdot \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} \right),$$

where $b \neq 0$, $a/b \in \mathbb{Q}$ and $f : T^2 \to [0, 1]$ is a smooth function vanishing at a single point $p \in T^2$. The chain recurrent set R is the whole torus, $R = T^2$, while

$$R_\xi = \begin{cases} T^2 & \text{if } \mu a + \nu b = 0, \\ f^{-1}(0) = \{p\} & \text{otherwise.} \end{cases}$$

Assuming, in addition, that $\mu a + \nu b \neq 0$, the set $C_\xi = T^2 - \{p\}$ is not closed. Nevertheless, a Lyapunov 1-form in the class $\xi \neq 0$ exists if and only if $\mu a + \nu b < 0$. In this case, $\omega = \mu\, dx + \nu\, dy$ is such a Lyapunov 1-form. This example shows that the existence of a Lyapunov 1-form for (Φ, R_ξ) does not imply C_ξ to be closed.

Example 3. Consider the standard irrational flow on the torus $X = T^2$, i.e. the flow of the vector field $V = a(\partial/\partial x) + b(\partial/\partial y)$, where $b \neq 0$ and $a/b \notin \mathbb{Q}$. Choose a cohomology class $\xi = \mu[dx] + \nu[dy] \in H^1(X; \mathbb{R})$ such that $\mu a + \nu b = 0$. Then $R = X$ and $R_\xi = \emptyset$ but condition (B) of Theorem 2 is not satisfied and so there is no Lyapunov 1-form for (Φ, R_ξ) in the class ξ.

This example shows that condition (B) is not a consequence of the fact that C_ξ is closed.

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